Quantum Hall physics in rotating Bose–Einstein condensates

TOPICAL REVIEW

Quantum Hall physics in rotating Bose–Einstein condensates

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Abstract
The close theoretical analogy between the physics of rapidly rotating atomic Bose condensates and the quantum Hall effect (i.e. a two-dimensional electron gas in a strong magnetic field) was first pointed out ten years ago. As a consequence of this analogy, a large number of strongly correlated quantum-Hall-type states have been predicted to occur in rotating Bose systems, and suggestions have been made for how to manipulate and observe their fractional quasiparticle excitations. Due to a very rapid development in experimental techniques over the past years, experiments on BEC now appear to be close to reaching the quantum Hall regime. This paper reviews the theoretical and experimental work done to date in exploring quantum Hall physics in cold bosonic gases. Future perspectives are discussed briefly, in particular the idea of exploiting some of these strongly correlated states in the context of topological quantum computing.

(Some figures in this article are in colour only in the electronic version)

Contents

1. Introduction

Almost a decade ago, it was realized [1] that there exists an intimate theoretical connection between two seemingly very different physical systems: the quantum Hall effect (QHE) [2], which occurs in a two-dimensional electron gas (2DEG) at very low temperatures and strong magnetic fields, and a rapidly rotating, dilute Bose–Einstein condensate [3] of electrically neutral atoms. The key to this analogy is the observation that in two dimensions rotation and a perpendicular magnetic field play a very similar role, making the two systems mathematically equivalent. This implies that, at sufficiently fast rotation, a Bose condensate is expected to enter a regime with strongly correlated states of the quantum Hall type, including quasiparticle excitations that obey fractional (anyon) statistics [4].

In recent years, the quantum Hall—rotating BEC analogy has been theoretically explored in great detail, starting with the prediction [5] of a bosonic Laughlin state [6]. In addition, a large number of other incompressible states are, in principle, expected to occur, including the bosonic analog of the Jain sequence [7] and other Abelian states, but also non-Abelian states [8, 9]. Moreover, a number of exotic states have been predicted in the case of rotating bosons with spin [10].

Unfortunately, experiments have not yet reached this quantum Hall regime. However, there is reason to be optimistic, as the experimental development over the past decade has been astounding. The first experiments on rotating atomic Bose condensates were performed in the late 1990s, and the first observation of a quantized vortex reported in 1999 [11, 12]. Since then, Abrikosov lattices with hundreds of vortices have been produced, and present-day experiments [13] are close to the rotation speed at which this vortex lattice is predicted to melt and the system enter the quantum Hall regime. The main obstacle is that at these rotation speeds the system is close to the point where the centrifugal potential cancels the external harmonic trap and the atomic cloud would fly apart. There are, however, recent proposals for how to get around this problem by modifying the confining potential, so that the quantum Hall regime may be reachable in the near future.

In addition to being interesting in its own right, the prospect of producing quantum-Hall-type states in cold atom systems may have long term practical applications. One of the reasons for the recently revived interest in the anyonic excitations of the QHE is the theoretical proposal to use them in the context of quantum computing [14]. This vision is certainly very far into the future. On the other hand, atomic systems may eventually turn out to be superior to the 2DEG quantum Hall system, as they allow for a very large degree of controlled tunability of various experimental parameters, including interaction strength and details of the confining potential. Moreover, these systems are well isolated and clean and thus less prone to decoherence than solid state realizations.

This paper presents a (hopefully) comprehensive review of this, still active, field of research. Section 2 gives a brief overview of the fractional quantum Hall effect. In section 3 we give a general introduction to the subject of rotating Bose condensates, summarizing the experimental developments of the past decade and the theoretical understanding of how the system goes from a vortex lattice to the quantum Hall regime as rotation is increased. Section 4 explains the theoretical equivalence between a fast-rotating Bose gas and electrons in a strong magnetic field, along with some of the basic properties of the resulting many-body spectrum in the presence of interactions. An account of the literature on Abelian bosonic quantum Hall states is given in section 5; most of this work is numerical or based on trial wavefunctions such as those of the composite fermion phenomenology [7] and involves testing for the occurrence and stability of incompressible states (and their fractional excitations) at, e.g., the Jain fractions. This section also contains a brief discussion of the applicability of the composite fermion scheme for very low-angular-momentum states. Quite some work has been done to study the possible occurrence of non-Abelian quantum Hall states, which are particularly interesting in the context of topological quantum computing. This is accounted for in section 6, which concludes that such states appear to be more prominent in rotating BEC than in the conventional QHE. Following this, we summarize various recent proposals for how to design experiments capable of reaching the quantum Hall regime with present-day experimental techniques (section 7).
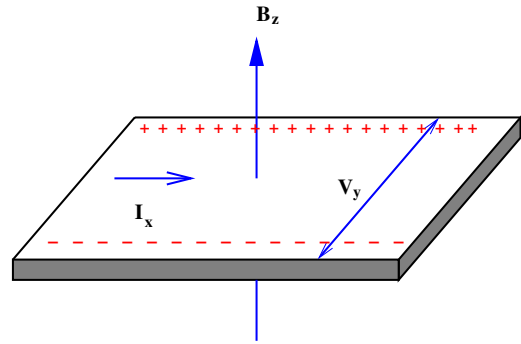

Figure 1. Sketch of the Hall experiment. The 2DEG is exposed to a strong perpendicular magnetic field B_z. A current I_x is passed through the sample along the x-direction, and the resulting transverse voltage V_y measured for varying values of the magnetic field.

Finally, we briefly discuss multicomponent Bose condensates (section 8) and round off with some concluding remarks and future perspectives in section 9.

2. The fractional quantum Hall effect—a brief overview

The fractional quantum Hall effect (FQHE) [2] is one of the most intriguing and most studied phenomena in condensed matter physics during the past two to three decades. It occurs in two-dimensional, high-mobility electron systems (typically formed at the interface between two semiconductor crystals, e.g. in GaAs heterostructures) subjected to a strong magnetic field and low temperatures (in the millikelvin regime in present-day experiments). Figure 1 shows a sketch of a typical Hall experiment: a current I_x is passed through the sample along the x-direction and the resulting transverse voltage V_y measured for varying values of the magnetic field. Roughly speaking, the occurrence of a transverse voltage can be understood as being due to the deflection of the charge carriers in the presence of the external magnetic field, causing a build-up, or imbalance, of charge along the edges of the sample. In a purely classical picture, this leads to a linear relation for the Hall resistance $R_{xy} = V_y/I_x$ as a function of the magnetic field B. In the *quantum* Hall effect, however, the Hall resistance is quantized,

$$R_{xy} = \frac{V_y}{I_x} = \frac{1}{\nu} \frac{h}{e^2}, \tag{1}$$

where h is Planck's constant and e the electron charge. The number ν takes integer values (integer quantum Hall effect) or is equal to rational fractions (fractional quantum Hall effect), and each allowed value of R_{xy} remains constant for a finite range of the magnetic field, as indicated in figure 2. At the same time, the longitudinal resistivity ρ_{xx} is equal to zero, except for the transition regions between neighboring plateaux. A third characteristic property of each of these quantum Hall states is that the system is *incompressible*, i.e. there is a gap between the ground state and (bulk) excitations. The integer effect was discovered in 1980 by von Klitzing *et al* [15]; two
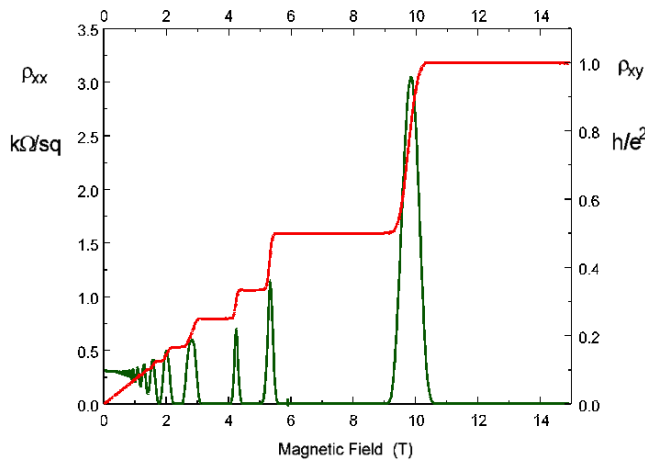
Figure 2. Sketch of the (integer) quantum Hall effect. The Hall resistance as function of the magnetic field is quantized, i.e. exhibits plateaux. (The result predicted by the *classical* Hall effect corresponds to a straight line through the centers of the plateaux.) The longitudinal resistivity is zero except at transitions between plateaux. Courtesy of DR Leadley, Warwick University 1997.

years later, using even cleaner samples, Tsui and collaborators reported the discovery of the fractional effect [16] at $\nu = 1/3$. Since then, with the fabrication of ever-higher mobility samples, a large number of fractions have been observed[1]. The quantization of the Hall resistance turned out to be extremely exact (to at least ten parts in a billion), which has led to the introduction of a new standard of resistance, with the so-called von Klitzing constant $R_K = h/e^2$, roughly equal to 25 812.8 Ω, as the fundamental unit.

Physically, the number ν in (1) corresponds to the Landau level filling fraction at the center of the corresponding plateau[2]. In other words, the IQHE occurs when an integer number of Landau levels is filled, while the FQHE is seen at fractional filling. As a consequence of this, the integer effect may be qualitatively understood in terms of a non-interacting electron picture [2]. The fractional effect, on the other hand, is a much more subtle phenomenon, taking place (mostly) in the lowest Landau level (LLL) and possible to understand only when the interactions among the electrons are taken into account. The theoretical explanation of the most prominent fractional states at $\nu = 1/m$, where m is an odd integer, was given by Laughlin in 1983, when he proposed his famous many-body wavefunction for the ground state [6],

$$\psi(z_1, z_2, \ldots, z_N) = \prod_{i<j}(z_i - z_j)^m e^{-\sum_i |z_i|^2/4}. \qquad (2)$$

Here, $z_i \equiv x_i + iy_i\sqrt{eB/\hbar c}$ are two-dimensional complex coordinates denoting the positions of the particles in the

plane. It makes intuitive sense that this wavefunction does a good job in minimizing the Coulomb energy of the strongly correlated electrons in the plane—it contains the mth power of a *Jastrow factor*, which is a product of the 'distances' (relative coordinates) between all pairs of particles. The Jastrow factor approaches zero when any two particles try to come close to one another and thus, in a sense, helps to keep the particles apart[3]. This is a useful picture to keep in mind, as this factor will show up in various contexts later in the article.

The Laughlin wavefunction describes a novel, inherently quantum mechanical state of matter, an *incompressible quantum fluid*. One of the most exotic properties of this state, and all other fractional quantum Hall states, is that it supports *fractional excitations*; Laughlin showed that its fundamental, charged quasiparticle excitations carry a fraction $(1/m)$ of the electron charge and obey *fractional (anyon) statistics* [4]. The latter means that they are neither bosons nor fermions; when two such quasiparticles are exchanged in a counterclockwise manner, their wavefunction picks up a phase

$$\psi(\mathbf{r}_2, \mathbf{r}_1) = e^{i\pi\alpha}\psi(\mathbf{r}_1, \mathbf{r}_2) \qquad (3)$$

with $\alpha = 1/m$ for the $\nu = 1/m$ state. (Bosons and fermions would correspond to $\alpha = 0$ and 1, respectively.) Laughlin's theoretical explanation of the FQHE earned him the 1998 Nobel prize in physics, together with the experimental discoverers Tsui and Störmer.

As mentioned previously, a large number of FQH states have been observed since the discovery of the $\nu = 1/3$ plateau. Most of these occur at odd-denominator fractions, and many (but not all) belong to the 'Jain sequences' $\nu = n/(2np \pm 1)$ with n and p integers. There have been two main theoretical approaches to these states. In the Haldane–Halperin hierarchy picture [17, 18], a QH state can give rise to a sequence of 'daughter states' as successive condensates of quasielectrons and/or quasiholes. The basic idea is that, once the system is in a Laughlin state and a sufficiently large number of quasielectrons or quasiholes have been generated (typically by changing the magnetic field away from its value at the center of the quantum Hall plateau), these quasiparticles themselves may form a strongly correlated state, in much the same way as the electrons form the Laughlin state. The result is a new incompressible QH ground state at a different filling fraction, whose quasiparticles may again condense to form the next 'daughter' etc. The other approach is based on Jain's phenomenology of *composite fermions* [7]. The main idea of this construction is, roughly speaking, to attach an even number of vortices to each electron. These vortices effectively cancel a part of the external magnetic field, thus mapping the electrons into weakly interacting *composite* fermions, which can then be thought of as moving in a reduced magnetic field. This picture provides a method to construct explicit trial many-body wavefunctions for the ground states at $\nu = n/(2np \pm 1)$, as well as their quasiparticle excitations. This approach has proven highly successful, producing wavefunctions with very

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[1] While high mobility is necessary in order for the FQHE to be observed, the effect would actually go away if there were no impurities in the system at all. The occurrence of FQH plateaux requires a small but essential amount of disorder.

[2] Landau levels are the quantized energy levels of charged particles in a magnetic field. The degeneracy N_s of each Landau level is equal to the number of flux quanta penetrating the system, and ν is the filling fraction N_e/N_s, where N_e is the number of electrons in the system. Some more details on Landau levels will be discussed in section 4.

[3] Note that a single power of the Jastrow factor would be sufficient to fulfill the requirement that any fermionic many-body wavefunction has to go to zero as any two particles approach each other. So the Laughlin wavefunction goes to zero faster than what is simply required by the Pauli principle.

high overlaps with the corresponding exact states; we will get back to the details in section 5.1. Recent work [19], based on the use of conformal field theory methods to construct hierarchical FQHE wavefunctions, illustrates that these two seemingly competing approaches are, in fact, very closely related [20, 21].

While almost all polarized FQH states observed to date occur at odd-denominator fractions, there is one known exception, namely, the gapped state at $\nu = 5/2$. It is believed to be described by the so-called *Pfaffian* wavefunction proposed by Moore and Read [8]. The Pfaffian is the exact ground state of a three-body repulsive interaction and describes a paired state very similar to a p-wave superconductor [22]; it is even more exotic than the states discussed above, in that its quasiparticle excitations obey *non-Abelian* fractional statistics [14]. This generalization of 'conventional' (Abelian) anyons requires a degenerate set of d states with quasiparticles at fixed positions $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$, such that an interchange of two quasiparticles i and j corresponds to a unitary operation in the subspace of these degenerate states,

$$\psi_\alpha \to \rho^{(ij)}_{\alpha\beta} \psi_\beta. \tag{4}$$

Here, $\rho^{(ij)}$ is a $d \times d$ unitary matrix, and the set $\{\psi_\alpha\}$ denotes an orthonormal basis of the degenerate states. If the unitary matrices corresponding to different quasiparticle interchanges do not commute, the particles are said to obey non-Abelian statistics. The recent theoretical proposal that it might be possible to use such non-Abelian anyons in the context of topological quantum computing [14] has spurred great interest in the physics of the $\nu = 5/2$ state. Moreover, there exist mathematical generalizations of the Pfaffian, the so-called Rezayi–Read (RR) or parafermion states [9]; although there are speculations that the recently observed QH plateau at $\nu = 12/5$ [23] might correspond to a $k = 3$ parafermion state, there is so far no unambiguous evidence of the existence of such, even more exotic, non-Abelian states in the quantum Hall system.

Direct analogies of all the above (and many more) features of the FQHE are, in principle, expected to occur in rapidly rotating Bose gases and have been extensively studied in the literature in recent years. After a general introduction to rotating Bose condensates in the next section, the remainder of this article will be devoted to discussing these analogies in more detail.

3. Rotating Bose condensates

Bose–Einstein condensation of magnetically trapped alkali atoms was first achieved in 1995 [24], opening up many new directions of research on the border between atomic and condensed matter physics. Soon after these seminal experiments, people became interested in the rotational properties of atomic Bose condensates, and the occurrence of quantized vortices [25] was predicted [26]. The subsequent experimental development has been astonishing. The first ever vortex in such an atomic cloud was reported by the JILA group in 1999 [11], and soon after by the Paris group [12]. In the
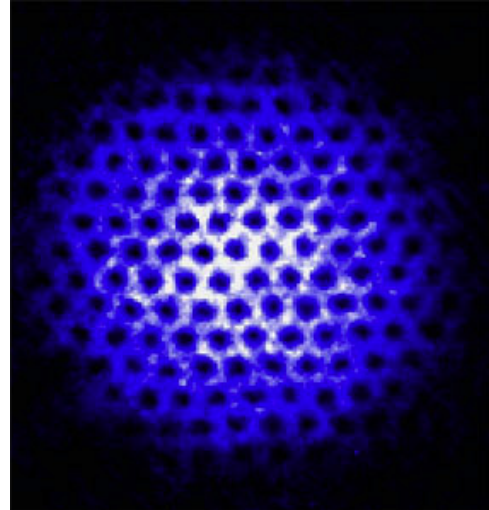

Figure 3. Image of an Abrikosov vortex lattice in a rotating BEC. From the JILA web page; courtesy of Eric Cornell.

former, the vortex state was obtained by a direct imprinting of the 2π phase shift onto the condensate, while the latter experiment used a mechanical stirring technique, with laser beams acting basically like a spoon in a cup of coffee [27]. Following these experiments, the same stirring technique was used to create ever larger numbers of vortices [28–30], which could be seen to organize themselves in triangular (Abrikosov) vortex lattices. A well established technique to visualize, e.g., such vortex arrays is to perform absorption imaging along the rotation axis—a picture is taken after switching off the trap and allowing the cloud to expand for a fraction of a second. The vortices then appear as density dips in the image, as shown in figure 3. There is a third method which can be applied to further increase the angular momentum once the cloud is rotating; it is based on 'evaporative spin-up', i.e. evaporating atoms with angular momentum smaller than average [31, 32]. Using this technique, it has been possible to create arrays with up to 200 vortices [33], and to study detailed properties such as the vortex modes [13, 34] and vortex cores [13, 33].

An interesting question to ask, then, is what will eventually happen to the vortex array if one keeps increasing the angular momentum of the system and thus the density of vortices. One might expect that at some point the vortex cores would start to overlap, as is the case in type II superconductors about to become normal. However, the picture that has emerged is quite different [35]. Since the confining potential in typical BEC experiments is harmonic, there is a limit where the centrifugal potential cancels the external potential and the cloud will become deconfined and fly apart (see section 4 for a more detailed discussion). When approaching this limit from below, the particles spread out, making the cloud more and more pancake shaped, and the effective interparticle interaction becomes weaker due to the decrease in density. As we shall see in the next section, these are preconditions for the BEC to be equivalent to a system of particles in the *lowest Landau level*. The entrance into the LLL regime is signaled by a shrinking of the vortex cores starting around the rotation frequency at which

the size of a vortex core becomes comparable to the spacing between vortices—the ratio between the size of a vortex core and the area occupied per vortex saturates to a constant. This behavior was predicted by Baym and Pethick [35, 36] and confirmed in recent experiments [13], where the transition to the LLL regime was observed to occur at around 98% of the deconfinement limit. Eventually, at even higher rotation, the vortex lattice is expected to melt. This melting transition has been studied theoretically by several groups [37–39] and is predicted to occur around $\nu \sim 6$–10, where the filling factor ν is the ratio of boson density to vortex density. Beyond this point, the system enters a regime of homogeneous, strongly correlated states of the same nature as those in the fractional quantum Hall effect.

So far, experiments have not actually reached this quantum Hall regime. Rotation frequencies of more than 99% of the deconfinement limit have been achieved [13], which is believed to be close to the vortex melting transition. The main practical problem is to push the rotation further without passing the point where the cloud flies apart. There are several very recent proposals of ways to avoid this problem, basically by modifying the external confinement. With these modifications, it may well be possible to reach the quantum Hall regime with presently available experimental techniques. We shall discuss these novel ideas in some more detail in section 7. Meanwhile, the next three sections summarize the theoretical analogies between rapidly rotating Bose condensates and the quantum Hall effect, simply assuming that the system is in the lowest-Landau-level regime.

4. Rotating bosons as a lowest-Landau-level problem

The basic insight, providing the analogy between rotating Bose condensates and the quantum Hall system, is that the Hamiltonian of a rotating system of harmonically confined, neutral particles is essentially equivalent to that of charged particles in an external magnetic field. We start by making this statement more precise and discussing under what circumstances a rotating Bose gas can be mapped to a lowest-Landau-level problem. The second part of this section presents some of the general properties of the corresponding many-body energy spectrum.

4.1. Mapping to the LLL

Let us consider a system of N spinless bosons with mass m in a harmonic trap of strength ω, rotating with angular frequency Ω and interacting via a short-range (delta function) potential H_I. In a rotating frame the Hamiltonian can be written as

$$H = \sum_{i=1}^{N} \left[\frac{\vec{p}_i^2}{2m} + \frac{1}{2}m\omega^2\vec{r}_i^2 \right] - \Omega L_z + H_I \qquad (5)$$

where L_z denotes the angular momentum around the rotation axis. (For simplicity, we will set $\hbar = 1$ whenever there is no risk of confusion.) Separating out the planar (x, y) part and completing the square inside the brackets, one can rewrite

equation (5) as

$$H = \sum_{i=1}^{N} \left[\frac{1}{2m}\left(\vec{p}_i - \vec{A}\right)_{\|}^2 + H_{\text{ho}}(z_i) \right] + (\omega - \Omega)L_z + H_I \quad (6)$$

with $\vec{A} = m\omega(-y, x)$, $\|$ denoting the planar (x, y) part of the Hamiltonian, while $H_{\text{ho}}(z)$ denotes the perpendicular (z) part of the harmonic oscillator potential. This is how the formal link to the quantum Hall system comes about: we see that the planar part of H takes the form of particles moving in an effective 'magnetic' field $\vec{B}_{\text{eff}} = \nabla \times \vec{A} = 2m\omega\hat{z}$. The quantum mechanical one-body spectrum of this part of the Hamiltonian is given by the so-called *Landau levels* (see, e.g., [2]), with energy $E_{n\|} = (n + \frac{1}{2})\hbar\omega_c$, where $n = 0, 1, 2, \dots$, and $\omega_c = 2\omega$. Each Landau level is degenerate in angular momentum, the number of states per Landau level being equal to the number of (effective) flux quanta piercing the plane. The single-particle wavefunctions in the symmetric gauge chosen here can be expressed as

$$\eta_{n,m} = N_{n,m}\, e^{-|z|^2/4}\, z^m L_n^m\left(\frac{z\bar{z}}{2}\right), \qquad (7)$$

where n is the Landau level index, m denotes the angular momentum, $N_{n,m}$ is a normalization factor, L_n^m are the associated Laguerre polynomials, and $z = \sqrt{2m\omega}(x + iy)$ is again a (dimensionless) complex coordinate denoting the particle position in the plane (note the change in notation as compared to (6)). Now, the interaction is assumed to be weak in the sense that it does not mix different harmonic oscillator levels. We will be interested, for a given total angular momentum, only in the *lowest-lying* many-body states (the 'yrast' band). In this limit, the model may be rewritten as a lowest-Landau-level (LLL) problem in the effective 'magnetic' field $B_{\text{eff}} = 2m\omega$ (and $n_z = 0$ for the harmonic oscillator in the z-direction). The Hamiltonian then reduces to the form

$$H = (\omega - \Omega)L + g\sum_{i<j}\delta^2(\mathbf{r}_i - \mathbf{r}_j) \qquad (8)$$

where L denotes the total angular momentum, $L = \sum_i l_i = L_z$. The single-particle states spanning our Hilbert space (the lowest Landau level) are thus (omitting normalization factors)

$$\eta_{0,m} = z^m e^{-\bar{z}z/4}. \qquad (9)$$

A general bosonic many-body wavefunction $\psi(z_1, \dots, z_N)$ with good angular momentum can thus be expressed as a homogeneous, symmetric polynomial in the coordinates $\{z_i\}$, times the exponential factor $\exp(-\sum_i |z_i|^2/4)$ (which will be suppressed throughout most of this paper for simplicity); the degree of the polynomial gives the total angular momentum of the state.

In the theoretical approach employed in the following sections, the system is simply assumed to be sufficiently dilute (i.e. weakly interacting) to be in the lowest Landau level at all angular momenta; the general strategy will be to look for the lowest (interaction) energy state within this subspace for given L and any fixed Ω. The experimental situation is somewhat different—we saw in section 3 that in order to
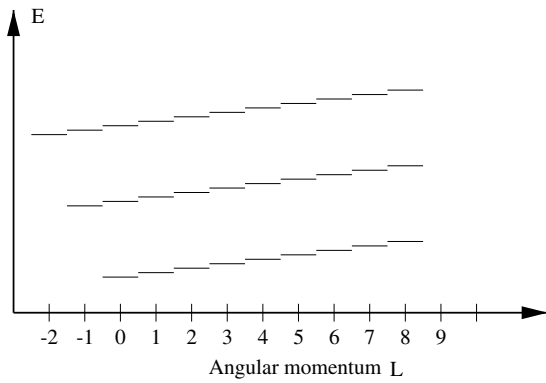
Figure 4. Sketch of the single-particle Landau level spectrum. The slope of each level is equal to $\omega - \Omega$.
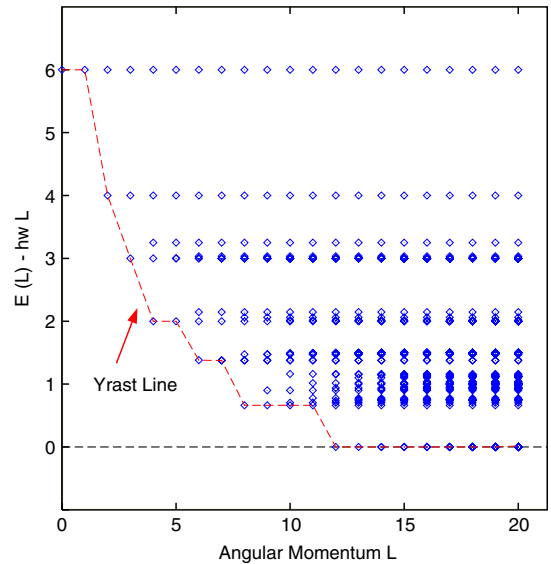

Figure 5. Yrast spectrum, i.e. many-body interaction energy as function of total angular momentum, for four bosons in a harmonic trap with delta function repulsion. This figure was first published in [41].

become sufficiently spread out to be in this dilute regime the cloud, in present-day experiments, has to rotate faster than $\Omega \approx 0.98\omega$. Moreover, a more natural picture in an experimental setting is to think of Ω as fixed, while the system selects the ground state angular momentum such as to minimize $E_I + (\omega - \Omega)L$. Then, in order to avoid Landau level mixing, the number of particles and/or $(\omega - \Omega)$ have to be small (see figure 4).

4.2. Yrast spectra

A convenient way of studying the many-body properties of a rotating boson system is to display its *yrast spectrum*[4], where the lowest many-body energy eigenvalues are plotted as a function of total angular momentum. An example is shown in figure 5, which was obtained by an exact diagonalization of the Hamiltonian (5) of the previous section for four particles, with the lowest-Landau-level restriction imposed. In the absence of interactions, the system is highly degenerate—the degeneracy for a given total angular momentum L corresponding to the number of ways L quanta of angular momentum may be distributed among N bosons. This degeneracy is lifted by the short-range repulsion, leading to the energy band seen in the figure (the spectrum is shown for $\Omega = \omega$, i.e. shows purely the interaction energy, cfr. equation (8)). The line connecting the lowest states at different angular momenta is commonly called the yrast line. A number of basic properties of the system may be read off this spectrum. First of all, one notices that the lowest possible energy decreases with increasing angular momentum, starting from $L = 0$, where all bosons sit in the lowest-angular-momentum state; this is due to the particles' ability to spread out more in the plane, as more and more angular momentum states become available to them (or, in other words, because the particles interact only for zero relative angular momentum). In particular, one notices that at $L = N(N-1) = 12$ and above the ground state has *zero* interaction energy. This is the first immediate consequence of the analogy

to quantum Hall physics: just as in the quantum Hall effect, it is possible to construct particularly well correlated states of the Jastrow form, i.e.

$$\psi(\{z_i\}) = \prod_{i<j} (z_i - z_j)^{2m} f(z) \tag{10}$$

where m is a positive integer, and $f(\{z_i\})$ is some homogeneous, symmetric polynomial in the N coordinates $\{z_i\}$. Since the repulsive interaction between the bosons is zero range, any state of this form will have zero interaction energy. Since the total power of a Jastrow factor $\prod_{i<j}(z_i - z_j)$ is equal to the number of pairs of particles, $N(N-1)/2$, the Jastrow part of the wavefunction (10) contributes an angular momentum $L_0 = mN(N-1)$. Therefore, the smallest angular momentum at which a state of the type (10) can exist is $N(N-1)$. At this angular momentum, the exact, non-degenerate ground state of the system is given by the *Bose–Laughlin state*

$$\psi(\{z_i\}) = \prod_{i<j} (z_i - z_j)^2 \tag{11}$$

as was first pointed out in [5]. The degeneracy of the zero-energy yrast states at $L > L_0$ corresponds to the number of ways $L - L_0$ angular momentum quanta can be distributed among the N particles; alternatively, this degeneracy can be found [41] by exploiting the fact that the wavefunctions (10) describe anyons in the lowest Landau level [42] obeying Haldane's exclusion statistics [43] with statistics parameter $g = 2$.

Another important feature of the yrast spectrum is that for each state there is a set of 'daughter states' at all higher L, with the same (interaction) energy. These are simply center-of-mass excitations of their 'parent' state [44]. At a given L, all states corresponding to center-of-mass excitations

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[4] 'Yrast' is a term traditionally used in nuclear physics when studying rotational spectra of nuclei. It is a Swedish word meaning 'most dizzy', i.e. the highest possible angular momentum at a given energy. In the context of rotating Bose condensates, the term 'yrast spectrum' was first introduced by Ben Mottelson [40]. Similar spectra have been used extensively in the quantum Hall literature.

of lower-L eigenstates are orthogonal to the subspace of 'new' states. According to Trugman and Kivelson [44], the latter subspace consists of *translation invariant* (TI) polynomials, i.e. polynomials which are invariant under a simultaneous, constant shift of all coordinates, $z_i \rightarrow z_i + a$. A convenient basis for the TI subspace is the elementary symmetric polynomials of degree L, $s_L(\{\tilde{z}_i\})$ ($2 \leqslant L \leqslant N$), defined as

$$s_L(\tilde{z}_1, \ldots, \tilde{z}_N) = \mathcal{S}\{\tilde{z}_1 \tilde{z}_2 \ldots \tilde{z}_L\} \qquad (12)$$

with $\tilde{z} \equiv z - Z$, where $Z = \sum_i z_i / N$ is the center of mass, and $\mathcal{S}$ denotes symmetrization of the product over all N particle coordinates. Together with $s_1(\{z_i\}) = z_1 + z_2 + \cdots + z_N = NZ$, the elementary polynomials (12) thus span the entire space of symmetric, homogeneous polynomials in the lowest Landau level. We will make explicit use of the basis (12) when discussing low-angular-momentum states in section 5.4.

The yrast spectra of rotating Bose condensates in the weak-interaction (lowest Landau level) regime have been studied extensively in the literature, using a variety of methods including analytical studies [40], mean field (Gross–Pitaevskii) theory [45], and exact numerical diagonalization [46–48]. For example, Reimann *et al* [47] demonstrated how the presence of localized vortices in a rotating boson cloud is revealed by periodic cusps in the yrast line of the exact many-body spectrum. However, the dominating line of attack in studying the analogies to quantum Hall physics has been the use of various types of trial wavefunctions. These include bosonic versions of Laughlin- and Jain-type wavefunctions [6, 7] and variations thereof [48, 49], as well as the bosonic counterparts of non-Abelian quantum Hall wavefunctions [8, 9]. The results of these trial wavefunction studies are reviewed in more detail in the following sections.

5. Abelian quantum Hall states

Most known fractional quantum Hall states are Abelian. These include the Laughlin and Jain states and more generally all states which, in the hierarchy picture, are generated by any sequence of quasielectron and quasihole condensates, as was discussed in section 2. The bosonic counterparts of the Laughlin and Jain states have been extensively studied in the literature, mainly numerically. Apart from exact diagonalization, a particularly widely used technique is the composite fermion approach; this line of work is reviewed in the first and main part of this section. In addition, we briefly discuss the anyonic quasiparticles of these states, the bosonic hierarchy, and the possibility to apply the composite fermion formalism at very low angular momenta.

5.1. The Jain sequence and composite fermions

We argued in the previous section that translation invariant (TI) states play a special role in the rotation spectra. A particularly useful scheme of constructing TI trial wavefunctions that has been widely exploited in quantum Hall physics comes from

the phenomenology of *composite fermions* (CF)[5]. Composite fermions were first introduced by Jain [7] and have proven very successful in describing FQH states, quantum dots in high magnetic fields [50], and, as we shall now discuss, highly rotational states of Bose condensates [5, 41, 51, 52]. In quantum Hall physics, the basic picture of Jain's construction is, roughly speaking, that an even number of vortices is bound to each electron. Each of these vortices effectively cancels one flux quantum of the external magnetic field, and the electrons are thus mapped into weakly interacting *composite* fermions, which can be thought of as moving in a reduced magnetic field. Technically, 'attaching a vortex' means multiplying the wavefunction by a Jastrow factor,

$$\prod_{i<j}(z_i - z_j). \qquad (13)$$

We have already mentioned that the Jastrow factor has the effect of keeping the particles apart—it goes to zero if any two coordinates z_i and z_j approach each other. Therefore, it takes care of much of the repulsive interaction between the particles. In the simplest approximation, the so-called non-interacting composite fermion (NICF) approach, the composite fermions are thus simply assumed to be non-interacting. This kind of considerations led Jain to construct trial wavefunctions a Slater determinant of (non-interacting) composite fermions in the reduced magnetic field, times an even power of Jastrow factors. In the case of bosons, whose wavefunction has to be symmetric rather than antisymmetric, the construction is modified by instead binding an *odd* number of vortices, mapping the bosons to weakly interacting composite fermions. In other words, bosonic trial wavefunctions with angular momentum L are constructed as non-interacting fermionic wavefunctions with angular momentum $L - pN(N-1)/2$, multiplied by an *odd* number p of Jastrow factors, and projected onto the LLL,

$$\psi_L = \mathcal{P}\left(f_S(z_i, \bar{z}_i)\prod_{i<j}(z_i - z_j)^p\right). \qquad (14)$$

Here, f_S denotes a Slater determinant consisting of single particle wavefunctions (7). The LLL projection $\mathcal{P}$ amounts to the replacement $\bar{z}_i \rightarrow 2\partial/\partial z_i$ in the polynomial part of the wavefunction—the recipe is to replace all the $\bar{z}$ with derivatives in the final polynomial, after multiplying out the Slater determinant and the Jastrow factors and moving all the $\bar{z}$ to the left. It has been shown [7] that with this projection method the single particle wavefunctions in the CF Slater determinant may be written as

$$\eta_{nl} = z^{n+l}\partial^n, \qquad l \geqslant -n \qquad (15)$$

with all derivatives acting only to the right. As this method tends to become computationally heavy in numerical calculations with many particles and a large number of derivatives, somewhat different methods of obtaining LLL

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[5] There is a class of wavefunctions within the CF formalism that are, by construction, translation invariant: they are called *compact states* and are characterized by having the nth CF Landau level occupied from $l_n = -n$ to $l_n = l_n^{\max}$ without any 'holes'.

wavefunctions have been employed in most of the CF literature [7]. These, too, are often referred to as projection. Nevertheless, in this paper, 'projection' will refer to the above 'brute force' procedure.

Before summarizing the results obtained in the literature, let us illustrate the method on two simple and well known examples in the QH regime: first, consider the case $L = N(N - 1)$. Taking $p = 1$, the Slater determinant f_S has to contribute an angular momentum $N(N - 1)/2$ and is given by putting all CFs into the lowest CF Landau level, from $l = 0$ to $N - 1$,

$$f_S = \begin{vmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_N \\ z_1^2 & z_2^2 & \dots & z_N^2 \\ \dots & \dots & \dots & \dots \\ z_1^{N-1} & z_2^{N-1} & \dots & z_N^{N-1} \end{vmatrix} \equiv \prod_{i<j}(z_i - z_j). \quad (16)$$

We immediately see from equation (14) that the full wavefunction is simply the bosonic Laughlin wavefunction (11) with angular momentum $L = N(N - 1)$. Next, consider the angular momentum $N(N - 1) - N$, corresponding to a 'quasiatom' (the bosonic counterpart of a quasielectron) at the center. While a quasihole can be seen as a local depletion of the quantum Hall liquid, a 'quasiatom' corresponds to a local contraction, with fractional surplus charge (or rather particle number in our case of neutral bosons) 1/2. In the CF language, a trial wavefunction for such a quasiparticle excitation at the center, i.e. with minimum angular momentum, is obtained by exciting one composite fermion to the second CF Landau level, leading to the Slater determinant

$$f_S = \begin{vmatrix} \bar{z}_1 & \bar{z}_2 & \dots & \bar{z}_N \\ 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_N \\ z_1^2 & z_2^2 & \dots & z_N^2 \\ \dots & \dots & \dots & \dots \\ z_1^{N-2} & z_2^{N-2} & \dots & z_N^{N-2} \end{vmatrix}. \quad (17)$$

We see that it is the excited composite fermion, with single particle wavefunction $\sim \bar{z}$ (to be replaced by a derivative upon LLL projection), that causes the reduction of angular momentum as compared to the Laughlin state. One obtains the full trial wavefunction (again, apart from the exponential factor)

$$\psi_{qp} = \sum_{i=1}^{N}(-1)^i \partial_i \prod_{k<l;k,l\neq i}(z_k - z_l) \prod_{m<n}(z_m - z_n)$$

$$\propto \sum_{i=1}^{N} \sum_{j\neq i} \frac{1}{z_i - z_j} \prod_{k\neq i}(z_i - z_k)^{-1} \psi_L \quad (18)$$

with ψ_L denoting the Laughlin state (11). This wavefunction has very high overlap with the exact one (e.g. 99.7% for four bosons [41]). Its fermionic counterpart has been proven to capture correctly both the fractional charge and the anyonic statistics of the QH quasielectron [53, 54], and the same can be expected to be the case for this bosonic version.

Trial wavefunctions for other yrast states are constructed in similar ways. The lower the angular momentum, the larger

the number of derivatives. Had we filled up the second CF Landau level, with equally many composite fermions as the first, we would have obtained a trial wavefunction for a new incompressible *ground state*, with filling fraction 2/3. In general, ground states of the *principal Jain sequence* $\nu = n/(n+1)$—the bosonic counterpart of the well known principal Jain sequence $\nu = n/(2n + 1)$ in the FQHE—are described as $\nu^* = n$ integer quantum Hall states of $p = 1$ composite fermions, while quasihole and 'quasiatom' excitations are described by removing a CF from a filled CF Landau level and adding a CF to an otherwise empty CF Landau level, respectively. Of course this construction can be generalized in the usual way by attaching a larger (odd) number $p = 3, 5, \dots$ of vortices to each boson. Note, however, that for a pure delta function interaction the ground states at the corresponding filling fractions $\nu = n/(np + 1)$ belong to the highly degenerate subspace of zero-energy states discussed in section 4.2, making this construction less relevant. Adding higher derivatives of the delta function to the Hamiltonian (or, equivalently, using Haldane's pseudopotentials [2]) lifts this degeneracy, and the CF construction with $p > 1$ again provides good wavefunctions for the resulting ground states [41].

The idea of applying the CF phenomenology in the context of rotating Bose gases was first tested by Cooper and Wilkin [5] and by Viefers *et al* [41] in disk geometry [2] for small numbers of particles, by comparing to exact diagonalization results. It was shown that the approach reproduces many prominent features of the yrast spectrum, such as the locations of cusps in the yrast line; overlaps with the exact solutions for a number of yrast states were computed for up to ten particles and shown to be large—typically 99% for five particles. Later, several more systematic studies were performed [51, 52] in spherical geometry [17]. The advantage of this theoretical approach, in which the particles move on the surface of a sphere with a radial magnetic field produced by a magnetic monopole at the center, is that the sphere has no boundaries. While edge effects play an important role for small systems in the plane, this geometry thus allows for the 'simulation' of homogeneous bulk states even for the relatively modest particle numbers accessible to numerical calculations. Instead of the complex coordinates z_i discussed so far, the particle positions on the sphere are parametrized by the polar angles (θ_i, ϕ_i), or more conveniently by the spinor coordinates

$$u_i = \cos(\theta_i/2)e^{i\phi_i/2}, \qquad v_i = \sin(\theta_i/2)e^{-i\phi_i/2}. \quad (19)$$

For example, the Bose–Laughlin wavefunction (11) takes the form

$$\psi = \prod_{i<j}(u_iv_j - u_jv_i)^2, \quad (20)$$

and other wavefunctions may be translated from the plane to the sphere in a similar way. Performing exact diagonalizations for up to 12 particles on the sphere, Regnault and Jolicoeur [51] found evidence of the occurrence of incompressible (gapped) states at the principal Jain fractions $\nu = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$, as well as excited states in general agreement with the CF phenomenology. Moreover, going away from pure delta function interaction by adding a higher-order pseudopotential

(as discussed above), they found evidence of an incompressible state at $\nu = 2/5$. This state is not part of the principal Jain sequence; rather, it is the bosonic counterpart of the 2/7-state in the FQHE. Provided it is Abelian, it can be understood, in the hierarchy picture, as resulting from a condensate of quasiholes in the $\nu = 1/2$ Bose–Laughlin state; in the CF picture, it belongs to the negative $p = 3$ Jain sequence, $\nu = n/(3n - 1)$, with $n = 2$. Although the interaction in typical experiments is dominated by s-wave scattering, there exist methods to introduce and enhance an additional d-wave interaction [55–57]. This may provide a possibility, at least in principle, to observe the 2/5 state (and other states with $\nu < 1/2$). Alternatively, one might use a system of atoms with permanent dipolar interaction [58] such as chromium [59]. Following up on the work of Regnault and Jolicoeur, Chang *et al* [52] performed a direct comparison between exact diagonalization results and those predicted from the CF approach, computing energies as well as overlaps between exact and CF wavefunctions for the ground states and low-lying excitations at $\nu = \frac{1}{2}$, $\frac{2}{3}$ and $\frac{3}{4}$. They found that the non-interacting composite fermion approach correctly predicts the incompressibility of the ground states at $\nu = \frac{1}{2}$, $\frac{2}{3}$ and $\frac{3}{4}$ and produces excellent overlaps (over 97% for up to 10 particles) for the ground state and excitations at $\nu = \frac{1}{2}$ as well as the ground state at $\nu = \frac{2}{3}$. However, for increasing n, the NICF approximation gets progressively worse, for short-range as well as Coulomb interaction, producing considerably poorer overlaps than in the principal Jain sequence of the electronic FQHE. In the CF language, the interpretation is that 'residual interactions' between the composite fermions play an important role. In particular, in the limit $n \to \infty$, i.e. $\nu = 1$ (the bosonic counterpart of the metallic $\nu = \frac{1}{2}$ state in the quantum Hall system), the ground state of the system can *not* be described by a Fermi sea but rather appears to be a non-Abelian state, the bosonic version of the Moore–Read Pfaffian [8]. We shall get back to this point in section 6.

Additional evidence for the strongly correlated nature of the states at $\nu = 1/2$, 2/3 and 3/4 was given by Cazalilla *et al* [60], who studied the low-energy edge excitations of harmonically confined, rapidly rotating few-boson systems. According to Wen [61], the 'topological order' of a bulk quantum Hall state, implying its filling fraction, as well as the charge and statistics of its quasiparticle excitations, is reflected in the properties of its edge excitations. Performing exact diagonalization studies for up to seven particles, Cazalilla *et al* showed that the number of edge modes is consistent with that predicted by Wen's theory, for the states at $\nu = 1/2$, 2/3 and 3/4.

5.2. Anyonic excitations

A particularly interesting aspect of the fractional quantum Hall effect is the existence of fractionally charged quasielectron- and quasihole excitations [6], which are expected to obey fractional (anyonic) statistics [4]. Obviously, the same type of quasiparticles should occur in the bosonic quantum Hall system, with 'charge' replaced by particle number. In the simplest case of the $\nu = 1/2$ Laughlin state, the quasihole

would be a vortex with local lack of density corresponding to half an atom, and would obey semionic statistics, i.e. quantum statistics 'halfway' between bosons and fermions. Such a quasihole, located at z_0, is described by the wavefunction [6]

$$\psi_{qh}(\{z_i\}) = \prod_i (z_i - z_0) \prod_{k<l} (z_k - z_l)^2. \quad (21)$$

Paredes *et al* [62] suggested that, in principle, such quasiholes can be created by piercing the Bose–Laughlin state locally with lasers. Adiabatically moving such a laser would then 'drag' the quasihole along, enabling controlled interchange of pairs of quasiholes and thus a direct measurement of the anyonic phase $\pi/2$ picked up under exchange. The latter would be particularly interesting—despite very promising recent experimental progress [63] in the electronic FQH system, a direct and unambiguous measurement of fractional statistics is still lacking.

5.3. Hierarchy

In addition to the Jain states discussed above, the analogy with the FQHE in principle predicts a large number of (Abelian) hierarchical states that do not belong to the principal Jain sequence [17, 18, 64]. An example of such a state in the 2DEG is the one recently observed at $\nu = 4/11$ [65], whose bosonic counterpart would be $\nu = 4/7$. Trial wavefunctions for these hierarchy states, or at least those corresponding to 'quasiatom' (as opposed to quasihole) condensates, may be constructed using conformal field theory techniques [19, 64, 66]; although this construction is well defined, its validity will eventually have to be determined by numerical tests of the resulting wavefunctions. As discussed above, even the principal Jain states do not describe the bosonic system as accurately as is the case in the 2DEG, and the same may be the case for the general hierarchy construction. Moreover, as we have seen, trial wavefunctions for ground states at $\nu < 1/2$ are of interest only in systems with scattering in higher partial waves.

5.4. Low angular momenta—a digression

Let us briefly address some interesting analytic results in a case that is very far away from the quantum Hall regime, namely the very lowest angular momenta up to the single vortex, $2 \leqslant L \leqslant N$. Within the lowest-Landau-level approximation, exact ground state wavefunctions for all angular momentum states in this interval were derived some years ago [67, 68]. They are given by the elementary symmetric polynomials $s_L(\tilde{z}_i)$ where $\tilde{z}_i = z_i - Z$ and $Z = \sum_i z_i / N$ is the center-of-mass coordinate,

$$\psi_L^{ex} = \sum_{p_1 < p_2 < \cdots < p_L} (z_{p_1} - Z)(z_{p_2} - Z) \cdots (z_{p_L} - Z). \quad (22)$$

For example, $\psi_{L=2} = \mathcal{S}[(z_1 - Z)(z_2 - Z)]$, with $\mathcal{S}$ denoting symmetrization over all particle coordinates; for $L = N$ this expression reduces to $\psi_{L=N} = \prod_i (z_i - Z)$. Since in present-day experiments the lowest-Landau-level approximation is certainly not valid at these lowest rotational states, the results discussed in this subsection may be somewhat academic.

However, there is an interesting connection between the exact wavefunctions (22) and those following from a naive application of the composite fermion construction. *A priori*, one would certainly expect the CF construction to fail in this regime, at least if the usual qualitative picture of composite fermions were to be taken literally. According to this picture, it is the flux attachment (i.e. the factor $\prod(z_i - z_j)^p$) which makes the composite fermions weakly interacting, justifying the NICF approach. Note, however, that since the Jastrow factor itself has an angular momentum of $mN(N-1)$, the construction of CF trial wavefunctions at $L \sim N$ involves applying $\mathcal{O}(N^2)$ derivatives [41, 69]. One would expect that these derivatives acting on the Jastrow factor destroy most of the good correlations which are at the very heart of the CF construction. It is therefore intriguing that in this regime a naive application of the NICF scheme produces wavefunctions whose overlap with the exact ground states (22) is not only large, but *increases* with increasing particle number. In [69] the single vortex state, $L = N$, was studied in detail, and numerical calculations for up to 43 particles showed that the overlap between the CF trial wavefunction and the exact analytical result (22) approaches unity for large N, with the difference decreasing as $\sim 1/N$ (see figure 6). Further analysis [70] showed that this is not an artifact of the $L = N$ state. Numerical tests up to $N = 7$ for $L = N - 1$ and up to $N = 8$ for $L = N - 2$ show the same tendency, i.e. overlaps increasing with particle number. (For fixed N, on the other hand, overlaps tend to decrease as L decreases.) In fact, the CF wavefunctions can be shown to have an analytical structure strikingly similar to the exact ones. The simplest example is the CF state at $L = N = 4$, which can be expressed as

$$\psi_{L=N=4}^{\mathrm{CF}} = \sum_{k=1}^{4} \prod_{i=1}^{4} \left(z_i - Z^{(k)} \right), \tag{23}$$

with $Z^{(k)}$ denoting the 'incomplete' center-of-mass coordinate $\sum_{j \neq k} z_j / (N - 1)$. The situation becomes more complicated for higher N and lower L, with more and more coordinates 'missing' in the center of mass (making the task of analytically proving the numerical results highly non-trivial), but apart from this, the general structure (22) is reproduced by the CF construction.

6. Non-Abelian quantum Hall states

Although the zoo of Abelian states discussed in the previous section is very rich, it does not exhaust all possibilities of gapped quantum Hall states. The possible existence of *non-Abelian* states in the quantum Hall system was already pointed out in section 2; in this section we shall discuss the bosonic analogies of these states. Non-Abelian quantum Hall states have received great attention lately, due to the recent proposal to use their quasiparticle excitations in the context of topological quantum computation [14]. The main advantage of this scheme is its intrinsic fault tolerance—quantum information stored in states with multiple non-Abelian quasiparticles is 'topologically protected', i.e. immune to local perturbations. To actually perform a quantum computation
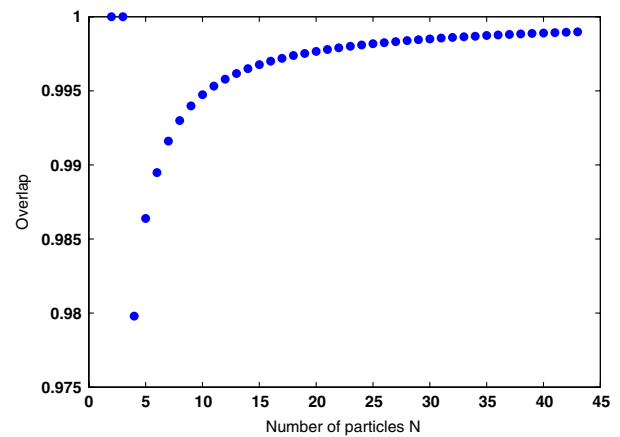

Figure 6. Overlap between the CF trial wavefunction at $L = N$ and the exact one given in equation (22), as a function of the number of particles. Note the range of numbers on the y-axis—the overlap is equal to 99.5% already for 10 particles.

would involve creating a state with a given number of quasiparticles at certain positions and performing controlled braiding operations by physically dragging quasiparticles around one another in a specified manner. One may thus speculate that, eventually, rotating Bose condensates may provide better candidates for topological quantum computing than the conventional QHE: in general, cold atomic systems are much easier to control and manipulate, with a high tunability of experimental parameters, and as already mentioned there exist theoretical proposals [62] of ways to drag quasiparticles through the condensate. Moreover, non-Abelian states may actually be more prominent in the bosonic case; numerical studies indicate that both the Pfaffian and other parafermion states can be expected to occur already in the lowest Landau level. So it is obviously of interest to study the possible occurrence of such states in the bosonic system. The next two subsections summarize the work that has been done in this direction.

6.1. The Pfaffian at $\nu = 1$

As was pointed out in section 5.1, $\nu = 1$ is the bosonic counterpart of the half-filled Landau level in the electronic FQHE. In the latter, there is no quantum Hall plateau around $\nu = 1/2$ [2, 71]; rather, the system displays a compressible state, behaving like a degenerate gas of (almost) free electrons in zero external magnetic field. This behavior has been explained [7, 72] in terms of the non-interacting composite fermion model—at $\nu = 1/2$ there are exactly two external flux quanta per electron, so after binding two flux quanta each, the composite fermions are left in zero effective field and form a Fermi sea. On the other hand, at $\nu = 5/2$, one does find an incompressible state which is believed to be described by the Moore–Read wavefunction, the quantum Hall analog of the paired state in a p-wave superconductor [22]; the picture is that the presence of the two filled lowest spin-subbands effectively modifies the interaction between the electrons in the half-filled topmost Landau level, leading to pairing.

In the bosonic system, the situation is qualitatively different, in the sense that there appears to be no compressible state at $\nu = 1$. Rather, there is quite substantial numerical evidence that the ground state corresponds to the bosonic version of the Moore–Read state,

$$\psi^{\mathrm{MR}}(\{z_i\}) = \prod_{i<j}(z_i - z_j)\,\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \qquad (24)$$

where the Pfaffian is defined as

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) = \mathcal{A}\left[\frac{1}{(z_1 - z_2)}\frac{1}{(z_3 - z_4)}\cdots\frac{1}{(z_{N-1} - z_N)}\right], \qquad (25)$$

with $\mathcal{A}$ denoting antisymmetrization over all coordinates. The possibility of a Pfaffian state at $\nu = 1$ was first suggested by Wilkin and Gunn [73]; work by the same group [37] later showed the existence of an incompressible ground state at $\nu = 1$, and reported large overlaps (more than 96% for six particles) between this state and the Pfaffian trial wavefunction; overlap calculations by Chang *et al* [52] for up to 16 particles confirm this picture. Further numerical evidence was given by Regnault and Jolicoeur [51, 74], both for the ground state (including evidence of pairing from the form of the two-particle correlation function) and for quasihole excitations (correct degeneracies and high wavefunction overlaps).

It appears clear that the existence of the Pfaffian state at $\nu = 1$ is not an artifact of the short-range interaction—on the contrary, introducing a Coulomb interaction between the bosons even increases the overlap [52]. (Though the introduction of a strong d-wave component in the interaction may destroy the state [51].) In other words, the fact that the bosons have a stronger tendency of pairing than their fermionic counterparts appears to be mainly due to their quantum statistics.

6.2. Parafermion states

An important difference between the system at hand and the 2DEG is that the bosonic system allows for states with $\nu > 1$ that are entirely in the lowest Landau level, due to the absence of Pauli blocking. It is thus of interest to understand what happens in the interval up to $\nu \approx 6$ where the system is expected to enter the Abrikosov vortex lattice regime [37]. It was first suggested by Cooper *et al* [37] that in this interval, at $\nu = k/2$; $k = 3, 4, 5, \ldots$ one may find a sequence of non-Abelian incompressible states described by the parafermion wavefunctions introduced by Read and Rezayi [9]. They can be represented as [75]

$$\psi^{(k)}(\{z_i\}) = \mathcal{S}\left[\prod_{i<j\in A}^{N/k}(z_i - z_j)^2 \prod_{k<l\in B}^{N/k}(z_k - z_l)^2 \ldots\right], \quad (26)$$

where the system is divided into k groups $(A, B, \ldots)$ each containing N/k particles, and $\mathcal{S}$ denotes symmetrization over all coordinates. The Laughlin state is recovered as the special case $k = 1$, while the expression for $k = 2$ is an equivalent way of writing the Pfaffian (24). Generalizing the pairing in the Moore–Read state, these wavefunctions describe states with k-particle clustering; they are the exact zero-energy eigenstates

of a $(k+1)$-particle delta function interaction. Performing numerical calculations on the torus, Cooper *et al* found large overlaps between the Read–Rezayi states (26) and the ground states at $\nu = k/2$ for $1 \leqslant k \leqslant 6$ and also recovered the correct ground state degeneracies. Regnault and Jolicoeur [51, 74] later took these calculations to larger systems (in spherical geometry) to see if this picture continues to hold as one approaches the thermodynamic limit. Their results remained somewhat inconclusive but indicated that for a pure delta function interaction, overlaps quickly decrease as k and the number of particles are increased. On the other hand it was shown by Rezayi *et al* [76] that the $\nu = 3/2$ parafermion state is stabilized by introducing a moderate amount of longer-range interaction; similar conclusions were reached for $\nu = 2$ in a very recent paper by Cooper and Rezayi [77]. In principle this may be achieved in a system with dipolar interactions or a moderate d-wave component. What makes these states particularly interesting is that $k = 3$ is the smallest k-value among the parafermion states for which the non-Abelian statistics support universal quantum computation [14].

7. Beyond harmonic potentials—ways to avoid the deconfinement problem?

We have seen that one of the main practical obstacles to actually reaching the quantum Hall regime experimentally, is that, for the usual harmonic confinement, rotation speeds exceedingly close to the deconfinement limit are required. The present record, with rotation at $\Omega > 0.99\,\omega$ [13], while having reached the lowest Landau level, still lies clearly within the Abrikosov lattice regime. An obvious way to be able to rotate the cloud faster than ω without it flying apart is to modify the confining potential. This section summarizes a few such proposals.

7.1. Quartic potentials

Several theoretical studies have addressed the effect of adding a small quartic term to the trap [35]. Among the predictions for the vortex array regime are the occurrence of singly quantized vortex arrays with a hole in the middle or, at very high rotation, a single, multiply quantized vortex at the center of the trap. In experiment, such an anharmonic trap has been created [27, 78] by applying a blue detuned laser propagating along the axis of the trap. This effectively amounts to adding a Gaussian potential $U(r) \sim U_0 \exp(-\alpha r^2)$ where r is the (planar) distance from the axis of rotation, and the constants U_0 and α are given by the parameters of the laser. For small αr^2 this potential is well approximated by quadratic (giving a small correction to the original harmonic trap) + quartic.

In a very recent numerical study, Morris and Feder [79] propose that using this type of quartic potential would make it possible to attain the Bose–Laughlin state (and other quantum Hall states) with currently accessible rotation rates. They show that the inclusion of such a potential tends to lower the critical rotation frequencies at which the quantum Hall states are expected to occur. Moreover, they predict that fine-tuning of the Gaussian parameters (depending on particle number and

interaction strength) is necessary in order to avoid destroying the Laughlin state, but that the required values of these parameters are within experimental reach [80]. In particular, the required experimental parameters should become more easily accessible if the number of particles in the condensate is reduced; this may be achieved [81] by adding a 1D optical lattice along the axis of rotation, splitting the condensate into an array of independent quasi-2D BECs.

7.2. Optical lattices

A somewhat different, very recent theoretical proposal involving optical lattices is due to Bhat *et al* [82]. They suggest to include a co-rotating optical lattice (in the tight binding regime) in addition to the harmonic potential, keeping the system confined even at critical rotation velocity $\Omega = \omega$. In addition to avoiding deconfinement of the atom cloud, this model displays intriguing physical properties: mapping the system to a Bose–Hubbard model, the authors show that the rotation introduces phase factors in the effective hopping term, $\hat{H}_{\mathrm{hop}} \sim \hat{a}_i^\dagger \hat{a}_j e^{-i\phi_{ij}} + \mathrm{h.c.}$, where the phase depends on the rotation velocity and particle mass as well as the lattice spacing. The linear response of the system to a potential gradient (tilt of the lattice) shows quantum-Hall-like features even for a single particle (and similarly for two particles). The authors give the following, qualitative explanation of the analogy to quantum Hall physics. The lattice, with its tunneling barriers, in some mean field sense mimics the repulsion experienced by a single particle from the rest of the 2DEG, with the inaccessible regions (maxima) of the lattice corresponding to the positions of the other electrons. Moreover, the phases picked up by a single particle when moving around the lattice simulate the effect of the correlation holes (or vortices) at these 'electron positions'. But clearly a lattice potential cannot support a liquid state, so the exact correspondence between this system and the FQHE remains to be fully clarified by further studies.

Finally, it is worth mentioning that there have been other theoretical proposals to create a quantum Hall effect for bosonic atoms, involving *non-rotating* optical lattices [83], were the magnetic field is simulated e.g. by means of laser-induced hopping [84]. A particular advantage of optical lattices is the extremely large degree of controllability, not only of the amount of flux per lattice cell but also the amount of disorder in the system.

All the proposals discussed here claim that the relevant model parameters are more or less within reach of present-day experimental techniques. Given these latest ideas, one can certainly hope for exciting experimental developments in the near future!

8. Multicomponent Bose condensates

So far we have restricted ourselves to single-component condensates of spinless (or scalar) bosons. The quantum Hall phenomenology we have discussed is thus analogous to the QHE of 'spinless' (fully polarized) electrons, i.e. the case where the effective Zeeman gap is sufficiently large that the

spin degree of freedom of the electrons is frozen out. Let us end with a brief discussion of the more general case where internal degrees of freedom, such as spin, play a role.

Polarization effects have been studied in the quantum Hall literature, and in particular it has been shown that under certain conditions the lowest energy charged excitation at $\nu = 1$ is a spin-textured object, a so-called skyrmion [85]. Moreover, there have been studies of spin polarization effects at the edge of a quantum Hall system, predicting the existence of spin-textured states for sufficiently smooth confining potentials [86, 87]. In the context of atomic Bose condensates, there are several ways of creating systems with internal degrees of freedom, promising an even richer phenomenology than in the quantum Hall system. One interesting approach to producing multicomponent Bose condensates is the simultaneous condensation of mixtures of different atomic isotopes such as ^{85}Rb and ^{87}Rb [88, 89]. Moreover, it is possible to create *spinor condensates* by trapping higher-spin atoms such as ^{87}Rb [90] or ^{23}Na [91, 92] in optical traps [93]. The advantage of this technique is that optical traps confine the atoms independently of their spin orientations—as opposed to traditional magnetic traps, which typically confine only one spin projection, effectively giving scalar condensates [90, 91].

Inspired by these experimental advances, Reijnders *et al* [10] have performed theoretical studies of rotating spin 1 condensates in the lowest-Landau-level approximation, predicting a rich phase diagram and a number of exotic states. In particular, in the quantum Hall regime, they predict several series of novel non-Abelian states which are generalizations of the Read–Rezayi states discussed in the previous subsection. One might expect that future studies will continue to reveal interesting new physics in high-rotation states of multicomponent Bose condensates.

9. Concluding remarks

A summary of the research on quantum Hall physics in rotating atomic gases is necessarily preliminary, as the field is still highly active. At the present stage it is probably fair to say that the theoretical side is well explored—there are many direct analogies to the conventional quantum Hall effect, but also physical differences, such as the expected occurrence of non-Abelian states in the lowest Landau level. The desirable next step would be for experiments to 'catch up' and reach the quantum Hall regime. There is reason to be optimistic: the experimental development has been rapid since the first theoretical prediction of a quantum Hall effect in rotating BEC and the first experimental creation of a quantized vortex in the late nineties. The race towards the first experimental realization of the bosonic quantum Hall regime is going on at the time of writing, hand in hand with new theoretical proposals for how to best design such an experiment.

One can only speculate about future developments. Given the large degree of controllability of various parameters in cold atom experiments, one may dream of the possibility to create and manipulate anyonic quasiparticles and directly measure their fractional statistics, Abelian or non-Abelian. This, in

turn, might be the first step towards implementing a topological quantum computer. But this certainly can not be expected to happen in the near future.

Acknowledgments

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